INVESTIGATION OF THE LOCAL AND AVERAGE ANGULAR RADIATION COEFFICIENTS FOR A PAIR OF CONCENTRIC CYLINDERS OF FINITE LENGTH

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Inzhenerno-Fizicheskii Zhurnal, Vol. 14, No. 6, pp. 1018–1024, 1968

UDC 536.3

Formulas are obtained for the local and average angular radiation coefficients for a pair of coaxial cylinders of finite length.

This paper is concerned with the determination of local and average angular radiation coefficients of a radiating system consisting of four bodies (zones), namely, a pair of concentric cylinders of finite length L (inner cylinder—body 1, outer cylinder—body 2) and two bases (bodies 3 and 4) in the form of parallel rings of width $\delta = r_2 - r_1$, where r_1 and r_2 are the radii of the inner and outer cylinders, respectively.

The investigation consists of an analysis of the equations of closure and reciprocity for the local and average angular radiation coefficients and the derivation of expressions for determining them and also includes a numerical calculation of the average angular radiation coefficients [1–5].

The investigation as a whole is systematic in character, since all the local and average angular radiation coefficients required for the subsequent numerical determination and investigation of the local radiative heat transfer characteristics of the radiating system are determined [6, 7].

It should also be noted that both the formulas obtained for the local and average angular radiation coefficients and the results of the numerical calculations based on these formulas were repeatedly checked on the basis of the closure equations for the local coefficients and the closure and reciprocity equations for the average coefficients.

In this investigation we have relied chiefly on the contour integration method proposed by Fok [3]. The available data relating to a radiating system of the type considered are disjoint and incomplete. In [5] only one expression is given in finite form, for the average angular coefficient of radiation from the surface of the outer cylinder onto itself (φ_{22}). The same source includes the data of an approximate numerical calculation for the average angular coefficient of radiation from the surface of the outer onto that of the inner cylinder. However, no expression in finite form for calculating this coefficient (φ_{21}) is given.

We have derived formulas in finite form for determining the nine principal local and five average angular radiation coefficients, including that for the coefficient φ_{21}.

Determination of the local angular radiation coefficients. The closure equations of the system are widely used in numerical calculations of the local angular radiation coefficients. In the given case of a radiating system consisting of four bodies (zones), three of which are nonconcave ($\varphi(M_1, F_1) = \varphi(M_3, F_3) = \varphi(M_4, F_4) = 0$), these equations take the following form:

$$\begin{gathered}\varphi(M_1, F_2) + \varphi(M_1, F_3) + \varphi(M_1, F_4) = 1,\\ \varphi(M_2, F_1) + \\ + \varphi(M_2, F_2) + \varphi(M_2, F_3) + \varphi(M_2, F_4) = 1,\\ \varphi(M_3, F_1) + \varphi(M_3, F_2) + \varphi(M_3, F_4) = 1,\\ \varphi(M_4, F_1) + \varphi(M_4, F_2) + \varphi(M_4, F_3) = 1. \end{gathered} \quad (1)$$

These four equations contain 13 different angular radiation coefficients, the calculation of which requires at least nine formulas for different local angular radiation coefficients. Then the remaining four local angular radiation coefficients can be determined on the basis of the closure equations. But in this case these equations can no longer be used for checking the numerical calculations. However, in view of the axial symmetry of the system, we can obtain several additional equations reflecting the symmetry conditions. These equations for like points M_3 and M_4 belonging, respectively, to surfaces F_3 and F_4 take the following form:

$$\begin{gathered}\varphi(M_3, F_1) = \varphi(M_4, F_1); \quad \varphi(M_3, F_2) = \varphi(M_4, F_2);\\ \varphi(M_3, F_4) = \varphi(M_4, F_3). \end{gathered} \quad (2)$$

On the basis of the three equations of (2), the number of local coefficients subject to direct determination can be reduced from nine to seven, but not to six, since by virtue of (2) the last two of the equations in (1) coincide.

Moreover, as a result of the reciprocity equation for ring elements on the surfaces of the inner and outer cylinders

$$\varphi(M_1, F_2)\, dF_1 = \varphi(M_2, F_1)\, dF_2, \quad (3)$$

which is valid for like points M_1 and M_2 belonging, respectively, to surfaces F_1 and F_2, we obtain the following relation:

$$r_1 \varphi(M_1, F_2) = r_2 \varphi(M_2, F_1), \quad (4)$$

since $dF_1 = 2\pi r_1 dl$; $dF_2 = 2\pi r_2 dl$.

Thus, the number of local angular coefficients subject to direct determination is reduced to six. However, to retain the possibility of checking the numerical calculations with the closure equations, expressions were obtained for the nine principal local angular radiation coefficients. In this case, all the closure equations can be used for checking the results of numerical calculations of the local angular coefficients based on the corresponding formulas for determining $\varphi(M_i, F_k)$, which, in their turn, can be checked on the basis of the closure equations.

Basic formulas for determining the local angular coefficients. These formulas, obtained chiefly by contour integration [3] and partly by the classical integral method [2], have the following form:

$$\varphi(M_2, F_1) = \frac{1}{\pi}\Big[P(M_2)\ \mathrm{arc\ tg}\ B(M_2) + P_1(M_2)\ \mathrm{arc\ tg}\ B_1(M_2) + n\Big(\mathrm{arc\ tg}\ C(M_2) + \mathrm{arc\ tg}\ C_1(M_2) - \frac{l}{2}\ \mathrm{arc\cos}\ n\Big)\Big], \quad (5)$$

$$\varphi(M_2, F_2) = 1 - \frac{1}{\pi}\Big[K(M_2)\ \mathrm{arc\ tg}\ A(M_2) + K_1(M_2)\ \mathrm{arc\ tg}\ A_1(M_2) + n\Big(\mathrm{arc\ tg}\frac{1}{2}C(M_2) + \mathrm{arc\ tg}\frac{1}{2}C_1(M_2) - l\ \mathrm{arc\ cos}\ n\Big)\Big], \quad (6)$$

$$\varphi(M_2, F_3) = \frac{1}{\pi}\Big[K(M_2)\ \mathrm{arc\ tg}\ A(M_2) - P(M_2)\ \mathrm{arc\ tg}\ B(M_2) - n\ \mathrm{arc\ tg}\ D(M_2) - \frac{l_{M_2}}{2}\ \mathrm{arc\ cos}\ n\Big], \quad (7)$$

$$\varphi(M_2, F_4) = \frac{1}{\pi}\Big[K_1(M_2)\ \mathrm{arc\ tg}\ A_1(M_2) - P_1(M_2)\ \mathrm{arc\ tg}\ B_1(M_2) - n\ \mathrm{arc\ tg}\ D_1(M_2) - \frac{l_{M_2}}{2}\ \mathrm{arc\ cos}\ n\Big], \quad (8)$$

$$\varphi(M_1, F_3) = \frac{1}{\pi}\Big[\mathrm{arc\ tg}\frac{1}{C(M_1)} + \frac{l_{M_1}}{2n}\ \mathrm{arc\ cos}\ n - \frac{P(M_1)}{n}\ \mathrm{arc\ tg}\ B(M_1)\Big], \quad (9)$$

$$\varphi(M_1, F_4) = \frac{1}{\pi}\Big[\mathrm{arc\ tg}\frac{1}{C_1(M_1)} + \frac{l_{M_1}}{2n}\ \mathrm{arc\ cos}\ n - \frac{P_1(M_1)}{n}\ \mathrm{arc\ tg}\ B_1(M_1)\Big], \quad (10)$$

$$\varphi(M_3, F_4) = \frac{1}{\pi}\Big[\frac{1}{2}\ \mathrm{arc\ cos}\ n + N(M_3)\ \mathrm{arc} \times \mathrm{tg}\ S(M_3) - I(M_3)\ \mathrm{arc\ tg}\ E(M_3)\Big], \quad (11)$$

$$\varphi(M_3, F_2) = \frac{1}{\pi}\left[I(M_3)\ \mathrm{arc\ tg}\ E(M_3) + \mathrm{arc\ tg}\ G(M_3)\right], \quad (12)$$

$$\varphi(M_3, F_1) = \frac{1}{\pi}\Big[\mathrm{arc\ tg}\ H(M_3) - N(M_3)\ \mathrm{arc\ tg}\ S(M_3)\Big], \quad (13)$$

where

$$A(M_2) = \sqrt{1 + \frac{4}{l^2_{M_2}}}\sqrt{\frac{1}{n^2} - 1};$$

$$C(M_i) = \frac{l_{M_i}}{\sqrt{1-n^2}} \quad (M_i \in F_i; \quad i = 1, 2);$$

$$D(M_2) = \frac{l_{M_2}\sqrt{1-n^2}}{2(1-n^2) + l^2_{M_2}};$$

$$B(M_i) = \sqrt{\frac{l^2_{M_i} + (1+n)^2}{l^2_{M_i} + (1-n)^2}}\sqrt{\frac{1+n}{1-n}};$$

$$K(M_2) = \frac{l^2_{M_2} + 2}{\sqrt{l^2_{M_2} + 4}};$$

$$P(M_i) = \frac{l_{M_i}(l^2_M + n^2 - 1)}{\sqrt{(l^2_{M_i} + n^2 + 1)^2 - 4n^2}};$$

$$E(M_3) = \sqrt{\frac{l^2_{M_3} + (1+\rho_{M_3})^2}{l^2_{M_3} + (1-\rho_{M_3})^2}} \times \frac{\sqrt{1-n}\sqrt{\rho_{M_3}+n} + \sqrt{1+n}\sqrt{\rho_{M_3}-n}}{\sqrt{1+n}\sqrt{\rho_{M_3}+n} - \sqrt{1-n}\sqrt{\rho_{M_3}-n}};$$

$$S(M_3) = \sqrt{\frac{l^2_{M_3} + (\rho_{M_3}+n)^2}{l^2_{M_3} + (\rho_{M_3}-n)^2}}\ \frac{1}{H(M_3)};$$

$$H(M_3) = \sqrt{\frac{\rho_{M_3}+n}{\rho_{M_3}-n}};$$

$$G(M_3) = \frac{\sqrt{1+n}\sqrt{\rho_{M_3}+n} + \sqrt{1-n}\sqrt{\rho_{M_3}-n}}{\sqrt{1-n}\sqrt{\rho_{M_3}+n} - \sqrt{1+n}\sqrt{\rho_{M_3}-n}};$$

$$I(M_3) = \frac{\rho^2_{M_3} + l^2 + 1}{\sqrt{(\rho^2_{M_3} + l^2 + 1)^2 - 4\rho^2_{M_3}}};$$

$$N(M_3) = \frac{\rho^2_{M_3} + l^2 - n^2}{\sqrt{(\rho^2_{M_3} + l^2 + n^2)^2 - 4n^2\rho^2_{M_3}}}.$$

The quantities $A_1(M_2)$, $B_1(M_i)$, $C_1(M_i)$, $D_1(M_2)$, $K_1(M_2)$, and $P_1(M_i)$ in formulas (5)–(10) are determined using expressions obtained, respectively, from the values for $A(M_2)$, $B(M_i)$, $C(M_i)$, $D(M_2)$, $K(M_2)$, and $P(M_i)$ by substituting $l - l_M$ for l_M.

The dimensionless parameters (n, l) of the system and the dimensionless coordinates of the point M, at which the emitting element dF is located, are defined as follows: $n = r_1/r_2$ is the ratio of the radii of the inner and outer cylinders; $l = L/r_2$ is the dimensionless length of the cylinders; L is the cylinder length; $l_M = z_M/r_2$ is the dimensionless variable coordinate of the point M along the generatrix of the cylinder; z_M is the variable coordinate of the point M reckoned along the generatrix from the end face 3; $\rho_M = x_M/r_2$ is the dimensionless variable coordinate of the point M along the radius of the end face; and x_M is the variable coordinate of the point M reckoned along the radius of the end face from its geometric center.

Determination of the average angular radiation coefficients. In the system in question, there are 13 average angular radiation coefficients subject to determination, related by the following four closure and six reciprocity equations:

$$\left.\begin{aligned} \varphi_{12}+\varphi_{13}+\varphi_{14}=1; \quad \varphi_{11}=0, \\ \varphi_{21}+\varphi_{22}+\varphi_{23}+\varphi_{24}=1, \\ \varphi_{31}+\varphi_{32}+\varphi_{34}=1; \quad \varphi_{33}=0, \\ \varphi_{41}+\varphi_{42}+\varphi_{43}=1; \quad \varphi_{44}=0, \end{aligned}\right\} \tag{14}$$

$$\begin{gathered} \varphi_{12}F_1=\varphi_{21}F_2; \quad \varphi_{13}F_1=\varphi_{31}F_3; \quad \varphi_{14}F_1=\varphi_{41}F_4; \\ \varphi_{23}F_2=\varphi_{32}F_3; \quad \varphi_{24}F_2=\varphi_{42}F_4; \quad \varphi_{34}F_3=\varphi_{43}F_4. \end{gathered} \tag{15}$$

Moreover, we have the following equations derived from the symmetry conditions: $\varphi_{14}=\varphi_{13}$; $\varphi_{24}=\varphi_{23}$; $\varphi_{31}=\varphi_{41}$, $\varphi_{42}=\varphi_{32}$, $\varphi_{34}=\varphi_{43}$, of which only the first two are independent, the rest being simple corollaries of the reciprocity equations. In fact, since $F_3=F_4$, from reciprocity equations of the form $\varphi_{13}F_1=\varphi_{31}F_3$, $\varphi_{14}F_1=\varphi_{41}F_4$; $\varphi_{23}F_2=\varphi_{32}F_3$, $\varphi_{24}F_2=\varphi_{42}F_4$ we obtain

$$\frac{\varphi_{13}}{\varphi_{14}}=\frac{\varphi_{31}}{\varphi_{41}}; \qquad \frac{\varphi_{23}}{\varphi_{24}}=\frac{\varphi_{32}}{\varphi_{42}}. \tag{16}$$

Since by symmetry $\varphi_{14}=\varphi_{13}$, $\varphi_{24}=\varphi_{23}$, from relations (16) we find $\varphi_{31}=\varphi_{41}$; $\varphi_{32}=\varphi_{42}$. Moreover, from the reciprocity equation $\varphi_{34}F_3=\varphi_{43}F_4$ it follows directly that $\varphi_{34}=\varphi_{43}$.

Thus, bearing in mind that for the system in question we have three closure equations* and six reciprocity equations, together with two independent equations which follow from the symmetry conditions, we conclude that to calculate all 13 average angular radiation coefficients, we must have available two computational expressions for the missing pair of the 13 coefficients.

Expressions for five different (principal) average angular radiation coefficients are presented below. Therefore, three closure or reciprocity equations are available for checking the numerical calculations.

The expressions for the average angular radiation coefficients obtained by the classical integral method [2] have the following form:

$$\varphi_{21}=\frac{1}{\pi}\left(2n \operatorname{arc\,tg} c+e \operatorname{arc\,tg} b-\right.$$
$$\left.-\frac{1-n^2}{l}\operatorname{arc\,tg} f-\frac{l}{2}\arc\cos n\right), \tag{17}$$

$$\varphi_{22}=1+\frac{1}{\pi}\left(l \arccos n-\sqrt{l^2+4}\times\right.$$
$$\left.\times \operatorname{arc\,tg} a-2n \operatorname{arc\,tg}\frac{c}{2}\right), \tag{18}$$

$$\varphi_{23}=\frac{1}{2\pi}\left(\sqrt{l^2+4}\operatorname{arc\,tg} a-e \operatorname{arc\,tg} b+\right.$$
$$\left.+\frac{1-n^2}{l}\operatorname{arc\,tg} f-2n \operatorname{arc\,tg} d-\frac{l}{2}\arccos n\right), \tag{19}$$

$$\varphi_{13}=\frac{1}{2}-\frac{1}{\pi}\left[\operatorname{arc\,tg} c+\frac{1}{2n}\left(e \operatorname{arc\,tg} b-\right.\right.$$
$$\left.\left.-\frac{1-n^2}{l}\operatorname{arc\,tg} f-\frac{l}{2}\arccos n\right)\right], \tag{20}$$

$$\varphi_{34}=1-\frac{2}{\pi}\operatorname{arc\,tg} f-$$
$$-\frac{l}{1-n^2}\left[n-\frac{2}{\pi}\left(2n \operatorname{arc\,tg} c+e \operatorname{arc\,tg} b-\right.\right.$$
$$\left.\left.-\frac{1}{2}\sqrt{l^2+4}\operatorname{arc\,tg} a-n \operatorname{arc\,tg}\frac{c}{2}\right)\right], \tag{21}$$

where

$$a=\sqrt{1+\frac{4}{l^2}}\sqrt{\frac{1}{n^2}-1}; \quad c=\frac{l}{\sqrt{1-n^2}};$$

$$b=\sqrt{\frac{l^2+(1+n)^2}{l^2+(1-n)^2}}\sqrt{\frac{1-n}{1+n}};$$

$$d=\frac{l\sqrt{1-n^2}}{2(1-n^2)+l^2};$$

$$e=\frac{1}{l}\sqrt{(l^2+n^2+1)^2-4n^2}; \quad f=\sqrt{\frac{1+n}{1-n}}.$$

The results of numerical calculations of the average angular radiation coefficients for values of the parameters $l=1, 2, 4$ and $n=0.01$; 0.1; 0.25; 0.5 are presented in the table.

In the limit as $n=r_1/r_2 \to 0$ the coefficients φ_{12}, φ_{13}, φ_{21}, and φ_{31} become meaningless, since in this case we arrive at the essentially different problem of radiative heat transfer in a radiating system composed of a cylinder of finite length consisting of three zones (two bases and a lateral surface).**

Our investigation can serve as a basis for a broader and more general research involving the determination of local and average angular radiation coefficients and the various energy characteristics of radiative heat transfer [6].

Numerical Values of the Average Angular Radiation Coefficients

l	n	φ_{ik}							
		φ_{12}	φ_{13}	φ_{21}	φ_{22}	φ_{23}	φ_{31}	φ_{32}	φ_{34}
1	0.01	0.503	0.249	0.005	0.379	0.308	0.005	0.616	0.379
	0.1	0.528	0.236	0.053	0.352	0.298	0.048	0.602	0.350
	0.25	0.574	0.213	0.143	0.308	0.275	0.114	0.587	0.299
	0.5	0.674	0.163	0.337	0.229	0.217	0.217	0.579	0.204
2	0.01	0.706	0.147	0.007	0.581	0.206	0.006	0.824	0.170
	0.1	0.726	0.137	0.073	0.536	0.196	0.055	0.792	0.153
	0.25	0.760	0.120	0.190	0.460	0.175	0.128	0.747	0.125
	0.5	0.826	0.087	0.413	0.329	0.129	0.232	0.688	0.080
4	0.01	0.845	0.078	0.008	0.757	0.118	0.006	0.944	0.050
	0.1	0.857	0.072	0.086	0.693	0.111	0.058	0.897	0.045
	0.25	0.875	0.063	0.219	0.587	0.097	0.134	0.828	0.038
	0.5	0.910	0.045	0.455	0.406	0.069	0.240	0.740	0.020

*By virtue of the symmetry conditions, there is no difference between the closure equations $\varphi_{31}+\varphi_{32}+\varphi_{34}=1$ and $\varphi_{41}+\varphi_{42}+\varphi_{43}=1$.

**This kind of problem was considered in detail in [7].

NOTATION

L is the length of the cylinders; r_1 and r_2 are the radii of the inner and outer cylinders, respectively; n and l are the dimensionless parameters of the radiating system; F_i is the area of the surface i; dF_i is the area element of the surface i; $\varphi(M_i, F_k)$ is the local angular coefficient of radiation from the area element dF_i at the point M_i onto the finite surface F_k; φ_{ik} is the average angular coefficient of radiation from the surface F_i onto the surface F_k.

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20 September 1967

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